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Infinitesimal canonical transformations of generalised Hamiltonian equations

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Abstract. The infinitesimal canonical transformations of generalised Hamiltonian equations are discussed in this paper. It is shown that for the generalised Hamiltonian equations $u_t = D\delta H/\delta u$, the infinitesimal canonical transformations are also the Noether transformations, and both the approach in the Hamiltonian formalism and the one in the Lagrangian formalism lead to the same conserved densities.

1. Introduction

Studies of symmetries and conservation laws of physical systems have always been one of the central topics in physics. As is well known, the symmetries and conservation laws are closely related to each other. In past years, with the discovery that an infinite number of conservation laws could be found for a great variety of nonlinear wave equations, there has been a new surge in the study of symmetries and conservation laws (see e.g. Kumei 1975, 1977, 1978, Ibragimov 1976, 1977, Ibragimov and Anderson 1976, Crampin 1977, Dodd and Bullough 1977, Olver 1977, Wadati 1978, Abellanas and Galindo 1979, 1981, Fuchssteiner 1979, Shadwick 1979, 1980, Tu and Qin 1979, 1981a, Fujimoto and Watanabe 1980, Guerrero and Martinez Alonso 1980, Tu 1980, 1981).

This paper is divided into two parts. In § 2 the notations for generalised Hamiltonian equations (Lax 1978, Martinez Alonso 1979) are briefly sketched, and then the definition of infinitesimal canonical transformations is extended from ordinary Hamiltonian equations to generalised ones. It is shown that the following important result (Goldstein 1951) remains valid in the generalised cases:

> the generator of an infinitesimal canonical transformation which leaves invariant the Hamiltonian is a conserved density. (1.1)

In § 3 the generalised Hamiltonian equations

$$u_t = D\delta H/\delta u \tag{1.2}$$

are considered and it is shown that this equation can be reduced, by setting $u = \phi_x$, to an Euler–Lagrange equation with a Lagrangian density L, and the connection between L and H is similar to the classical Legendre transformation. Furthermore, the infinitesimal canonical transformation with generator G is also shown to be a Noether transformation and the same conserved density follows from both approaches, that is, from proposition (1.1) in the Hamiltonian formalism or from the Noether theorem in

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the Lagrangian formalism. It may be noted that in spite of the continuous progress of the two approaches connecting conserved densities with symmetries, there are few works, at least to the author's knowledge, revealing the relation between the two approaches, and we hope the result of this paper could be extended to the more general Hamiltonian equation $u_t = J\delta H/\delta u$. In § 3 there is an open problem in this connection.

2. Generalised Hamiltonian equations and infinitesimal canonical transformations

As is well known, the ordinary classical Hamiltonian equation (for continuous media) reads

$$\partial p_i/\partial t = -\delta H/\delta q_i, \qquad \partial q_i/\partial t = \delta H/\delta p_i, \qquad i = 1, \ldots, n, \tag{2.1}$$

where $\delta/\delta p$ and $\delta/\delta q$ denote the variational derivatives (Coelho de Souza and Rodrigues 1969, Kruskal *et al* 1970, Gel'fand and Dikii 1975, Galindo and Martinez Alonso 1978, Aldersley 1979, Tu and Qin 1979, 1981a, Tu 1980). Setting $\boldsymbol{u} = (p_1, \ldots, p_n, q_1, \ldots, q_n)^{\mathrm{T}}$ and

$$\delta/\delta \boldsymbol{u} = \left(\frac{\delta}{\delta p_1}, \ldots, \frac{\delta}{\delta q_n}\right)^{\mathrm{T}}, \qquad J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where T represents the transpose, and I_n is the identity matrix of order n, we may rewrite equation (2.1) in the concise form

$$\boldsymbol{u}_t = J_n \delta H/\delta \boldsymbol{u}. \tag{2.2$'$}$$

In this ordinary case the Poisson bracket of equation (2.1) is defined by

$$\{F, G\} = \sum_i \left(\frac{\delta F}{\delta q_i}\frac{\delta G}{\delta p_i} - \frac{\delta F}{\delta p_i}\frac{\delta G}{\delta q_i}\right),$$

which can be written as

$$\{F, G\} = (\delta F/\delta \boldsymbol{u})^{\mathrm{T}} J_n (\delta G/\delta \boldsymbol{u}). \tag{2.3$'$}$$

Definition 1. (Lax 1978, Martinez Alonso 1979, Tu 1980) The equation

$$\boldsymbol{u}_t = J\delta H/\delta \boldsymbol{u} \tag{2.2}$$

is called a generalised Hamiltonian equation if the operator J is linear and anti-symmetric, i.e. $(J\boldsymbol{u})^{\mathrm{T}}\boldsymbol{v} \overset{D}{=} -\boldsymbol{u}^{\mathrm{T}}(J\boldsymbol{v})$, where $\boldsymbol{u} = (u^1, \ldots, u^M)^{\mathrm{T}}$, $u^i = u^i(x_1, \ldots, x_N, t)$, $\delta/\delta \boldsymbol{u} = (\delta/\delta u^1, \ldots, \delta/\delta u^M)^{\mathrm{T}}$ and

$$\frac{\delta}{\delta u^r} = \sum_{ki_1\ldots i_k} (-1)^k D_{i_1} \ldots D_{i_k} \frac{\partial}{\partial u^r_{i_1\ldots i_k}}, \qquad u^r_{i_1\ldots i_k} = \frac{\partial^k u^r}{\partial x_{i_1} \ldots \partial x_{i_k}},$$

$$D_i = \frac{\partial}{\partial x_i} + \sum_{rki_1\ldots i_k} u^r_{i_1\ldots i_k i} \frac{\partial}{\partial u^r_{i_1\ldots i_k}},$$

and $f \overset{D}{=} g$ means $f - g = \Sigma_i D_i h_i$ for some vector $\boldsymbol{h} = (h_i, \ldots, h_N)^{\mathrm{T}}$. The Poisson bracket of the generalised Hamiltonian equations (2.2) is defined by an equation similar to equation (2.3)$'$:

$$\{F, G\} = (\delta F/\delta \boldsymbol{u})^{\mathrm{T}} J (\delta G/\delta \boldsymbol{u}). \tag{2.3}$$

A scalar function $f = f(u, u^{(1)}, \ldots, u^{(p)})$, which depends on $u(x, t)$ and its space derivatives $u^{(k)} = \{u^r_{i_1 \ldots i_k}\}$, is called a conserved density of equation (2.2) if $\mathrm{d}f/\mathrm{d}t \overset{D}{\simeq} 0$ holds when $u(x, t)$ is taken to be the solution of equation (2.2).

Note that here and sometimes below we work up to equivalence with respect to D as in, for example, Dodd and Bullough (1977). This kind of calculation enables us to deduce in a pure algebraic manner, and is also reasonable because under some appropriate conditions, such as $u(x, t)$ and all its space derivatives vanishing at the boundary, an equation $g \overset{D}{\simeq} 0$ would be equivalent to $\int g \, \mathrm{d}x = 0$. Thus in this case a conserved density f will correspond to a constant of motion (or 'first integral') $I = \int f \, \mathrm{d}x = \text{constant}$. Likewise the equation $(J\boldsymbol{u})^{\mathrm{T}}\boldsymbol{v} \overset{D}{\simeq} -\boldsymbol{u}^{\mathrm{T}}(J\boldsymbol{v})$ would be the same as $(J\boldsymbol{u}, \boldsymbol{v}) = -(\boldsymbol{u}, J\boldsymbol{v})$ with $(\boldsymbol{u}, \boldsymbol{v})$ being the inner product $(\boldsymbol{u}, \boldsymbol{v}) = \int \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} \, \mathrm{d}x$. The following formula (integrate by parts) will be frequently used in this connection: $f(Dg) \overset{D}{\simeq} -g(Df)$, where $D = D_i$ or $D = \mathrm{d}/\mathrm{d}x$ in the case of one space dimension.

Some typical examples of the generalised Hamiltonian equations are as follows.

Korteweg–de Vries (KdV) equation $u_t = uu_x + u_{xxx}$.

$$J = D = \mathrm{d}/\mathrm{d}x, \qquad H = u^3/6 - u_x^2/2.$$

Modified KdV equation $u_t = u^2 u_x + u_{xxx}$.

$$J = D, \qquad H = u^4/12 - u_x^2/2.$$

Sine-Gordon equation $q_{tt} - q_{xx} = \sin q$.

$$J = J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad H = (q_x^2 + p^2)/2 - \cos q, \qquad (p = q_t, \boldsymbol{u} = (q, p)^{\mathrm{T}}).$$

Sine-Gordon equation (in the light-cone coordinate) $u_{xt} = \sin u$.

$$J = D^{-1}, \qquad H = -\cos u.$$

Nonlinear Schrödinger equation $q_t = \mathrm{i}(q_{xx} + |q|^2 q_x)$.

$$J = J_1, \qquad H = \mathrm{i}(q^2p^2/2 - q_x p_x)$$

$(p = q^*$ the complex conjugate of q, $\boldsymbol{u} = (p, q)^{\mathrm{T}})$.

Boussinesq equation (Tu 1981) $q_{tt} = q_{xx} + 3(q_x^2)_x + q_{xxxx}$.

$$J = J_1, \qquad H = (p^2 + q_x^2 + 2q_x^3 - q_{xx}^2)/2, \qquad (p = q_t, \boldsymbol{u} = (q, p)^{\mathrm{T}}).$$

Regular long wave equation (Lax 1978) $u_t = u_x + uu_x + u_{txx}$.

$$J = D(1 - D^2)^{-1}, \qquad H = u^2/2 + u_x^2/2.$$

In ordinary classical mechanics, we call the infinitesimal transformation

$$\boldsymbol{u}' = \boldsymbol{u} + \varepsilon\boldsymbol{\eta}, \tag{2.4}$$

where $\boldsymbol{u} = (p_1, \ldots, q_n)^{\mathrm{T}}$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_{2n})^{\mathrm{T}}$, a canonical or symmetric transformation of equation (2.1), if equation (2.1) remains form invariant under the transformation. In particular, if in the case of a discrete system there exists a G such that

$$p_i' = p_i - \varepsilon\,\partial G/\partial q_i, \qquad q_i' = q_i + \varepsilon\,\partial G/\partial p_i, \tag{2.5}$$

then G is called a generator of the transformation (2.5) (Goldstein 1951). In the case of continuous media we may use $\delta G/\delta q_i$ and $\delta G/\delta p_i$ instead of $\partial G/\partial q_i$ and $\partial G/\partial p_i$ respectively; then equation (2.5) can be written as

$$\boldsymbol{u}' = \boldsymbol{u} + \varepsilon J_n \delta G/\delta \boldsymbol{u}. \tag{2.6$'$}$$

In the case of the generalised Hamiltonian equation (2.2) the similar equation

$$\boldsymbol{u}' = \boldsymbol{u} + \varepsilon J \delta G/\delta \boldsymbol{u} \tag{2.6}$$

is taken as the definition of infinitesimal canonical transformations if equation (2.2) remains form invariant under this transformation, in which case G will likewise be called a generator.

We have proved the following (Tu and Qin 1979, Tu 1980).

Theorem A. If G is a conserved density of the generalised Hamiltonian equation (2.2), then the infinitesimal transformation (2.6) is canonical or equivalently symmetric. The converse is true if J is invertible.

Note that when $J = J_n$, as in ordinary cases, the operator J is certainly invertible. However, J may be taken to be other operators, not necessarily matrix, such as $J = D = \mathrm{d}/\mathrm{d}x$, which can be also made invertible (Lax 1968, Olver 1977), if we restrict the domain of D to be functions $F(u)$ with $F(0) = 0$, since then $DF(u) = 0$ would imply $F(u) = 0$. Therefore in these two most important cases we have:

Theorem A$'$

G is a conserved density of equation (2.2)	$\Leftrightarrow$	(2.6) is canonical, or equivalently symmetric transformation of equation (2.2).

Let (Tu 1980)

$$\partial_{r;i_1\ldots i_k} = \partial/\partial u^r_{i_1\ldots i_k}$$

and for $\mathbf{f} = (f^1, \ldots, f^M)^{\mathrm{T}}$, set

$$V(f) = \begin{pmatrix} V_1(f^1) & \cdots & V_M(f^1) \\ \vdots & & \\ V_1(f^M) & & V_M(f^M) \end{pmatrix},$$

$$V_r(f^s) = \sum_{ki_1\ldots i_k} (\partial_{r;i_1\ldots i_k} f^s) D_{i_k} \ldots D_{i_k}.$$

Here and below, notations such as F, G, f, $\boldsymbol{g}$, etc all denote smooth (scalar or vector) functions of $\boldsymbol{u}$ and its space derivatives and these functions will be assumed to be implicitly dependent on $x_1, \ldots, x_N$ and t.

It is easy to see from the definition that for $\boldsymbol{u} = \boldsymbol{u}(t, x, \varepsilon)$

$$\mathrm{d}f/\mathrm{d}\varepsilon = V(f)\mathrm{d}\boldsymbol{u}/\mathrm{d}\varepsilon. \tag{2.7}$$

In particular

$$(\mathrm{d}/\mathrm{d}\varepsilon) f(\boldsymbol{u} + \varepsilon \boldsymbol{\eta})|_{\varepsilon=0} = V(f)\boldsymbol{\eta} \tag{2.8}$$

and for solutions $\boldsymbol{u}$ of equation (2.2),

$$f_t = V(f)J\delta H/\delta \boldsymbol{u}. \tag{2.9}$$

Furthermore, for scalar functions $f = f(\boldsymbol{u})$ and vector $\boldsymbol{g}$ we have

$$V(f)\boldsymbol{g} \overset{D}{\simeq} (\delta f/\delta \boldsymbol{u})^{\mathrm{T}}\boldsymbol{g}. \tag{2.10}$$

From equations (2.9) and (2.10) we find that

$$f_t = V(f)J\delta H/\delta \boldsymbol{u} \overset{D}{\simeq} (\delta f/\delta \boldsymbol{u})^{\mathrm{T}} J\delta H/\delta \boldsymbol{u} \equiv \{f, H\} \tag{2.11}$$

holds for solutions $\boldsymbol{u}$ of equation (2.2), and from equations (2.8) and (2.10) the variation δF of a function $F(\boldsymbol{u})$ under the infinitesimal transformation (2.6) will be

$$\begin{aligned} \delta F &\equiv F(\boldsymbol{u} + \varepsilon J\delta G/\delta \boldsymbol{u}) - F(\boldsymbol{u}) \\ &= \varepsilon[(\mathrm{d}/\mathrm{d}\varepsilon)F(\boldsymbol{u} + \varepsilon J\delta G/\delta \boldsymbol{u})]|_{\varepsilon=0} = \varepsilon V[F(\boldsymbol{u})]J\delta G/\delta \boldsymbol{u} \\ &\overset{D}{\simeq} \varepsilon(\delta F/\delta \boldsymbol{u})^{\mathrm{T}} J\delta G/\delta \boldsymbol{u} = \varepsilon\{F, G\}. \end{aligned} \tag{2.12}$$

From theorem A and equations (2.11) and (2.12) we deduce:

Proposition 1. Suppose that J is invertible; then the Hamiltonian H of the generalised equation (2.2) remains invariant under the infinitesimal canonical transformation (2.6), that is,

$$\delta H = H(\boldsymbol{u}') - H(\boldsymbol{u}) \overset{D}{\simeq} 0. \tag{2.13}$$

Here and below we shall follow the usual convention that equations such as $f(\boldsymbol{u}') = 0$, $g(\boldsymbol{u}') \overset{D}{\simeq} 0$, which involve the infinitesimal transformation (2.6), will be considered true up to order $\mathrm{O}(\varepsilon)$. Hence $f(\boldsymbol{u}') = 0$ means $f(\boldsymbol{u}') = \mathrm{O}(\varepsilon^2)$, and (2.13) means $\delta H \overset{D}{\simeq} \mathrm{O}(\varepsilon^2)$ and so on.

Proof. From equation (2.12) we have $\delta H \overset{D}{\simeq} \varepsilon\{H, G\}$, and by theorem A and the hypothesis, G is a conserved density of equation (2.2), that is, $G_t \overset{D}{\simeq} 0$, hence by equation (2.11) it holds that $\{H, G\} \overset{D}{\simeq} 0$, and consequently $\delta H \overset{D}{\simeq} 0$ as desired.

Note that the equations (2.11), (2.12) and (2.13) are generalisations of the corresponding results in ordinary classical mechanics.

3. Canonical transformation as Noether transformation

In this section we consider the generalised Hamiltonian equations

$$u_t = D\delta H/\delta u, \tag{3.1}$$

where $u = u(x, t)$ is a scalar function of a one-dimensional space variable x and time variable t, and $D = \mathrm{d}/\mathrm{d}x$. The well known KdV and modified KdV equations and the related higher-order families (Lax 1968, Olver 1977, Chern and Peng 1979) can all be written as equation (3.1). Other equations, such as the Kodemtzev–Petriashvili equation (Zakharov and Schulman 1980), also belong to this type.

The corresponding infinitesimal canonical transformations of equation (3.1) are

$$u' = u + \varepsilon D\delta G/\delta u. \tag{3.2}$$

By theorem A′, to every such a canonical transformation, there is a related conserved density G.

An open problem. We have proved (Tu 1979) that for the higher-order KdV and modified equations, all the infinitesimal symmetry transformations $u' = u + \varepsilon\eta$, with η being polynomials of $u_i = D^i u$, take the form of (3.2). Does this hold for the more general equations

$$u_t = u_{2n+1} + D\delta f(u, \ldots, u_p)/\delta u, \tag{3.3}$$

where $p < n$?

It is easy to prove that (see, e.g. Tu and Qin 1981b) equation (3.1) in its original form cannot be written as an Euler–Lagrange equation of some variational problem. However, a simple substitution $u = \phi_x$ is sufficient to this end.

Proposition 2. Set $u = \phi_x$ and

$$L = \phi_x\phi_t/2 - H(\phi_x). \tag{3.4}$$

Then

$$u_t = D\delta H/\delta u \Leftrightarrow \phi_{xt} = -\delta H(\phi_x)/\delta\phi \Leftrightarrow \hat{\delta}L/\delta\phi = 0,$$

where the variational derivative $\hat{\delta}/\delta\phi$ is taken with respect to both x and t, that is,

$$\frac{\hat{\delta}}{\delta\phi} = \sum_{ki_1\ldots i_k} (-1)^k D_{i_1} \ldots D_{i_k} \frac{\partial}{\partial\phi_{i_1\ldots i_k}}, \qquad \phi_{i_1\ldots i_k} = \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}},$$

with $i_r = 0$ or 1, and $x_0 = t$, $x_1 = x$.

Proof. Note that for $H = H(u) = H(\phi_x)$, it holds that

$$\delta H/\delta\phi = \partial H/\partial\phi - D\delta H/\delta\phi_x = -D\frac{\delta H(\phi_x)}{\delta\phi_x},$$

hence by setting $u = \phi_x$ in equation (3.1) we obtain $\phi_{xt} = D\delta H(\phi_x)/\delta\phi_x = -\delta H(\phi_x)/\delta\phi$, that is, $u_t = D\delta H/\delta u \Leftrightarrow \phi_{xt} = -\delta H/\delta\phi$. Furthermore $\hat{\delta}L/\delta\phi = \hat{\delta}/\delta\phi(\phi_x\phi_t/2) - \hat{\delta}/\delta\phi H(\phi_x) = -\phi_{xt} - \delta H/\delta\phi$ and thus $\hat{\delta}/\delta\phi = L = 0 \Leftrightarrow \phi_{xt} = -\delta H/\delta\phi$, which completes the proof.

Equation (3.4) is quite similar, by setting $q = \phi$, $p = \phi_x$, to the Legendre transformation in ordinary classical mechanics, which reads

$$L = \sum p_i \partial q_i/\partial t - H.$$

Thus, equation (3.4) may be viewed as a Legendre transformation of the generalised Hamiltonian equation (3.1).

After the substitution $u = \phi_x$ and $u' = \phi'_x$, the infinitesimal canonical transformation (3.2) is reduced to

$$\phi'_x = \phi_x + \varepsilon D\delta G(\phi_x)/\delta\phi_x \tag{3.5}$$

or $\phi' = \phi + \varepsilon\delta G/\delta\phi_x +$constant. Here, as in Wadati (1978), we assume that $\phi \to$ constant (independent of t), as $|x| \to \infty$. Hence

$$\phi'_t = \phi_t + \varepsilon(\delta G(\phi_x)/\delta\phi_x)_t. \tag{3.6}$$

Now we recall briefly the definition of a Noether transformation and the classical Noether theorem (Hill 1951). Consider the infinitesimal transformation

$$x' = x + \varepsilon\xi, \qquad t' = t + \varepsilon\tau, \qquad \phi' = \phi + \varepsilon\eta, \tag{3.7}$$

and let L be a Lagrangian density corresponding to some Euler–Lagrange equation. If the variation δL of L under the transformation (3.7) is a divergence,

$$\delta L \equiv L(x', t', \phi') - L(x, t, \phi) = \varepsilon(\Omega_t + \Gamma_x),$$

then the transformation (3.7) is called a Noether transformation. According to the Noether theorem, conserved densities can be constructed in connection with the infinitesimal Noether transformations. Moreover, Steudel (1975) has pointed out that one further simplification could be made in so doing, that is, the general transformation (3.7) can be reduced to

$$x' = x, \qquad t' = t, \qquad \phi' = \phi + \varepsilon\eta. \tag{3.8}$$

In this simple case the Noether theorem reads

$$T = (\partial L/\partial\phi_t)\eta - \Omega \overset{D}{\simeq} 0. \tag{3.9}$$

Proposition 3. If (3.2) is an infinitesimal canonical transformation of equation (3.1), then (3.5) is an infinitesimal Noether transformation of the equation

$$\phi_{xt} = -\delta H(\phi_x)/\delta\phi. \tag{3.10}$$

Moreover, the conserved density associated with this transformation according to the Noether theorem is just G.

Proof. Setting $\delta L \equiv L(\phi') - L(\phi)$, we have

$$\delta L = (\phi'_x\phi'_t - \phi_x\phi_t)/2 - [H(\phi'_x) - H(\phi_x)].$$

Now by proposition 1 $\delta H = H(\phi'_x) - H(\phi_x) \overset{D}{\simeq} 0$, and

$$\begin{aligned}
&(\phi'_x\phi'_t - \phi_x\phi_t)/2 \\
&\quad = [\phi_x + \varepsilon(\delta G/\delta\phi_x)_x][\phi_t + \varepsilon(\delta G/\delta\phi_x)_t]/2 - \phi_x\phi_t/2 \\
&\quad = \varepsilon((\delta G/\delta\phi_x)_x\phi_t) + [(\delta G/\delta\phi_x)_t\phi_x]/2 \\
&\quad = \varepsilon[((\delta G/\delta\phi_x)\phi_t)_x + ((\delta G/\delta\phi_x)\phi_x)_t - 2(\delta G/\delta\phi_x)\phi_{xt}]/2 \\
&\quad \overset{D}{\simeq} \varepsilon[((\delta G/\delta\phi_x)\phi_x)_t/2 - (\delta G/\delta\phi_x)\phi_{xt}] \\
&\quad \overset{D}{\simeq} \varepsilon[((\delta G/\delta\phi_x)\phi_x)_t/2 - \textstyle\sum (\partial G/\partial\phi_{i+1})D^i\phi_{xt}] \qquad (\phi_i = D^i\phi) \\
&\quad = \varepsilon(((\delta G/\delta\phi_x)\phi_x)/2 - G)_t,
\end{aligned}$$

which shows that the transformation (3.5) is indeed a Noether transformation; moreover, the corresponding conserved density according to equation (3.9) is

$$\begin{aligned}
T = \phi_x\delta\phi/2 - \Omega &= [\phi_x(\delta G/\delta\phi_x)/2 + \text{constant } \phi_x] - [\phi_x(\delta G/\delta\phi_x)/2 - G] \\
&= G + \text{constant } \phi_x \overset{D}{\simeq} G.
\end{aligned}$$

Two conserved densities will be usually considered the same, if they differ from each other only by a term of total x-derivatives. The proof is thus complete.

Proposition 4. Let $v = v(u, u_1, \ldots, u_p)$ and $\partial_j = \partial/\partial u_j$, $\partial'_j = \partial/\partial v_j$, where as above $u_i = D^i u$ and $v_i = D^i v$; then

$$\frac{\delta}{\delta u} = \sum_s \left[\left(\frac{\delta}{\delta u}\right)_s v\right](-D)^s \frac{\delta}{\delta v}, \tag{3.11}$$

where

$$\left(\frac{\delta}{\delta u}\right)_s = \sum \binom{r}{s}(-D)^{r-s}\partial_r$$

with $\binom{r}{s}$ denoting the binomial coefficients (Kruskal *et al* 1970, Galindo and Martinez Alonso 1978, Aldersley 1979, Tu and Qin 1981b).

Proof. From the identity $\partial_i D^j = \Sigma \binom{j}{k} D^{j-k}\partial_{i-k}$ (Kruskal *et al* 1970), we have

$$\frac{\delta}{\delta u} = \sum_i (-D)^i \partial_i = \sum_{ij} (-D)^i \frac{\partial(D^j v)}{\partial u_i}\partial'_j$$

$$= \sum_{ijk} (-D)^i \binom{j}{k} (D^{j-k}\partial_{i-k} v)\partial'_j = \sum_{ijr} (-D)^i \binom{j}{i-r} (D^{j-i+r}\partial_r v)\partial'_j$$

$$= \sum_{ijrk} (-1)^i \binom{j}{i-r}\binom{i}{k} (D^{(j-i+r)-(i-k)}\partial_r v) D^k \partial'_j.$$

Making use of the identity

$$\sum_i (-1)^i \binom{j}{i}\binom{i+r}{k} = (-1)^j \binom{r}{k-j},$$

we deduce that

$$\frac{\delta}{\delta u} = \sum_{jrk} (-1)^{r+j} \binom{r}{k-j} (D^{r-(k-j)}\partial_r v) D^k \partial'_j$$

$$= \sum_{sj} (-1)^{j+s}\left[\left(\frac{\delta}{\delta u}\right)_s v\right] D^{j+s}\partial'_j = \sum_s \left[\left(\frac{\delta}{\delta u}\right)_s v\right](-D)^s \frac{\delta}{\delta v}.$$

The proof is complete.

This proposition is useful in relating two different canonical transformations of generalised Hamiltonian equations. For example, the KdV equation $v_t = vv_1 + v_3$ and the MKdV equation $u_t = u^2 u_1 + u_3$ are related to each other by the well known Miura transformation $v = u^2 + \mu u_1$, $\mu = \sqrt{-6}$, and from the above proposition it is easy to deduce that $\delta/\delta u = (2u - \mu D)\delta/\delta v$, from which we can obtain that

$$(2u + \mu D) D\delta/\delta u = 6R_K(v) D\delta/\delta v \tag{3.12}$$

where $R_K(v) = D^2 + 2u/3 + u_1 D^{-1}/3$ is the recursion operator of the KdV equation (see Olver 1977). Equation (3.12) can then be used to establish a definite relation $6f_M^{(n)}(u) = f_K^{(n-1)}(v)$ between the canonical transformations of the KdV equation $v' = v + \varepsilon D\delta f_K^{(n)}(v)/\delta v$ and those of the MKdV equation $u' = u + \varepsilon D\delta f_M^{(n)}(u)/\delta u$, where $D\delta f_K^{(n)}(v)/\delta v = (R_K(v))^n v_1$ and $D\delta f_M^{(n)}(u)/\delta u = (R_M(u))^n u_1$ with $R_M(u) = D^2 + 2u^2/3 + 2u_1 D^{-1} u/3$ (Olver 1977). The details of the above and related calculations will be published elsewhere.

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